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# A QUEUEING SYSTEM WITH CATASTROPHE, STATE DEPENDENT INPUT PARAMETER AND ENVIRONMENTAL CHANGE

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## *ABSTRACT*

*In this paper, a finite capacity queueing system with state dependent input parameter operating in different environments with catastrophes is studied. The input parameter is a function of n, the number of customers present in the system. The input rate increases (decreases) according as n, the number of units in the system, is less (greater) than N, a pre- assigned number. We undertake the transient analysis of a limited capacity queueing system with two environmental states in the presence of catastrophes. Transient state solution is obtained by using the technique of probability generating function. The steady state results of the model are obtained by using the property of Laplace transform. Finally, some particular cases of the queuing model are also derived and discussed.*

***KEYWORDS:*** *Transient Analysis, Catastrophes, Environment, Input Parameter, Probability Generating Function, Laplace Transforms*

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## 1. INTRODUCTION

The notion of catastrophe played a very important role in various areas of science and technology, in particular birth and death queueing models. In recent years, the attention has been focused to study the queueing systems on certain extensions that include the effect of catastrophes. This consists of adding to the standard assumptions the hypothesis that the number of customers is instantly reset to zero at certain random times. The catastrophes occur at the service- facility as a Poisson process with rate $\xi$. Whenever a catastrophe occurs at the system, all the customers there are destroyed immediately, the server gets inactivated momentarily, and the server is ready for service when a new arrival occurs.

In this connection, a special reference may be made to the paper by Crescenzo, A. Di et al. [7]. Crescenzo, A. Di et al. [7] proved that the M/M/1 catastrophized processes may be suitable to approach a current hot topic of great biological relevance, concerning the interaction between myosin heads and actin filaments that is responsible for force generation during muscle contraction. However, the force of contraction may rise on changing other conditions like a change in temperature or pH or a slight stretching of the fiber. Now, in the present paper, we have added another factor of environmental change, i.e. the change in the environment affects the state of the queueing system. In other words, the state of the queueing system is a function of environmental change factors.

A large number of research papers have appeared dealing with population processes under the influence of catastrophes (see. e.g., Bartoszynski, R. et al. [1], Brockwell, P.J. [2] and Brockwell, P.J. et al. [3]). These works are also concerned with various quantities of interest, such as transition probabilities, the stationary probabilities and the time to extinction. It is also well known that computer networks with a virus may be modeled by queueing networks with catastrophes [4]. Jain, N.K. and Kanethia, D.K. [9] discussed and obtained the transient analysis of a queue with environmental and catastrophic effects. Liu, Youxin and Liu, Liwei [17] studied the transient probabilities of an M/PH/1 queue model with catastrophes which is regarded as a generalization of an M/M/1 queue model with catastrophes.

The layout of this paper is as follows. In section 2, we present the assumptions and definitions of the model. The detailed analysis of the main model is done in section 3. In section 4 & 5, some particular cases and the steady- state solution of the queueing model are also derived and discussed. Mean queue length and applications of the model are discussed in section 6 & 7.

## 2. ASSUMPTIONS AND DEFINITIONS

(i) The customers arrive in the system one by one in accordance with a Poisson process at a single service station. The arrival pattern is non- homogeneous i.e., there may exist two arrival rates, namely 0 and $\lambda 1(n)$ [n= 0, 1, 2, M, where M denotes the size of the waiting space], of which only one arrival rate is operative at any instant.

(ii) The Poisson arrival rate $\lambda 1(n)$ is assumed to depend on the number (say n) waiting in the queue, including the one in service in such a manner that whenever this number is equal to a fixed number (say N) we have some normal rate as $\lambda 1$ and that for number of units greater than N the rate is lower and for the number of units less than N it is higher than the normal rate. We therefore define,

$$\lambda_1(n)=\lambda_1\left[1+\varepsilon(N-n)\right] \text{ with } n\leq N+\frac{1}{\varepsilon}$$

and $$0\leq n\leq N+\frac{1}{\varepsilon}\leq M$$

where $\varepsilon$ is a positive number such that $\varepsilon\geq\frac{1}{M-N}$. This restriction on M is necessary to avoid a negative value of $\lambda 1(n)$, the arrival intensity. When n=N or $\varepsilon$ =0, $\lambda 1(n)$ gives the normal rate as $\lambda 1$.

(iii) The customers are served one by one at the single service channel. The service times are independent identically exponentially distributed random variables. Further, it has been assumed that the system has service rates $\mu 1$ and $\mu 2$ corresponding to arrival rates $\lambda 1(n)$ and 0 respectively. The state of the queueing system when operating with arrival rate $\lambda 1(n)$ and service rate $\mu 1$ is designated as E whereas the other with arrival rate 0 and service rate $\mu 2$ is designated as F.

(iv) The Poisson rates at which the system moves from environmental states F to E and E to F are denoted by $\alpha$ and $\beta$ respectively.

(v) When the system is not empty, catastrophes occur according to a Poisson process with rate ξ. The effect of each catastrophe is to make the queue instantly empty; simultaneously, the system becomes ready to accept the new customers.

(vi) The queue discipline is first- come- first- served.

(vii) The capacity of the system is limited to M. i.e., if at any instant there are M units in the queue then the units arriving at that instant will not be permitted to join the queue, they will be considered lost for the system. Define,

Pn (t) = Joint probability that at time t the system is in state E and n units are in the queue, including the one in service.

Qn(t) = Joint probability that at time t the system is in state F and n units are in the queue, including the one in service.

Rn(t) = The probability that at time t there are n units in the queue, including the one in service. Obviously,

Rn(t) = Pn(t) + Qn(t)

Let us reckon time t from the instant when there are zero customers in the queue and the system is in the environmental state E so that initially, we have

$$P_n(0) = \begin{cases} 1 & ; \quad n = 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Qn(0) = 0 ; for all n.

## 3. FORMULATION OF MODEL AND ANALYSIS (TIME DEPENDENT SOLUTION)

The differential- difference equations governing the system are:

$$\frac{d}{dt}P_0(t) = -(\beta + \xi + \lambda_1(0))P_0(t) + \mu_1 P_1(t) + \alpha Q_0(t) + \xi \sum_{n=0}^{M} P_n(t); \quad n = 0 \tag{1}$$

$$\frac{d}{dt}P_n(t) = -(\lambda_1(n) + \mu_1 + \beta + \xi)P_n(t) + \mu_1 P_{n+1}(t) + \lambda_1(n-1)P_{n-1}(t) + \alpha Q_n(t); \quad 0 < n < M \tag{2}$$

$$\frac{d}{dt}P_M(t) = -(\mu_1 + \beta + \xi)P_M(t) + \lambda_1(M-1)P_{M-1}(t) + \alpha Q_M(t); \quad n = M \tag{3}$$

$$\frac{d}{dt}Q_0(t) = -(\alpha + \xi)Q_0(t) + \mu_2 Q_1(t) + \beta P_0(t) + \xi \sum_{n=0}^{M} Q_n(t); \quad n = 0 \tag{4}$$

$$\frac{d}{dt}Q_n(t) = -(\mu_2 + \alpha + \xi)Q_n(t) + \mu_2 Q_{n+1}(t) + \beta P_n(t); \quad 0 < n < M \tag{5}$$

$$\frac{d}{dt}Q_M(t) = -(\mu_2 + \alpha + \xi)Q_M(t) + \beta P_M(t)\,; \qquad n = M \tag{6}$$

Define, the Laplace Transform by

$$\text{L.T. } [f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s) \tag{7}$$

Now, taking the Laplace transform of equations (1)–(6) and using the initial condition, we get

$$(s + \beta + \xi + \lambda_1(0))\bar{P}_0(s) - 1 = \mu_1 \bar{P}_1(s) + \alpha\, \bar{Q}_0(s) + \xi \sum_{n=0}^{M} \bar{P}_n(s) \tag{8}$$

$$(s + \lambda_1(n) + \mu_1 + \beta + \xi)\bar{P}_n(s) = \mu_1 \bar{P}_{n+1}(s) + \lambda_1(n-1)\bar{P}_{n-1}(s) + \alpha\, \bar{Q}_n(s) \;;\; 0<n<M \tag{9}$$

$$(s + \mu_1 + \beta + \xi)\bar{P}_M(s) = \lambda_1 (M-1)\bar{P}_{M-1}(s) + \alpha\, \bar{Q}_M(s) \tag{10}$$

$$(s + \alpha + \xi)\bar{Q}_0(s) = \mu_2\, \bar{Q}_1(s) + \beta\, \bar{P}_0(s) + \xi \sum_{n=0}^{M} \bar{Q}_n(s) \tag{11}$$

$$(s + \mu_2 + \alpha + \xi)\bar{Q}_n(s) = \mu_2\, \bar{Q}_{n+1}(s) + \beta\, \bar{P}_n(s)\,; \qquad 0<n<M \tag{12}$$

$$(s + \mu_2 + \alpha + \xi)\bar{Q}_M(s) = \beta\, \bar{P}_M(s) \tag{13}$$

Define, the probability generating functions by

$$P(z,s) = \sum_{n=0}^{M} \bar{P}_n(s) z^n \tag{14}$$

$$Q(z,s) = \sum_{n=0}^{M} \bar{Q}_n(s) z^n \tag{15}$$

$$R(z,s) = \sum_{n=0}^{M} \bar{R}_n(s) z^n \tag{16}$$

where

$$\bar{R}_n(s) = \bar{P}_n(s) + \bar{Q}_n(s)$$

Multiplying equations (8)–(10) by the suitable powers of z, summing over all n and using equations (14)–(16), we have.

$$\lambda_1 \varepsilon z^2 (z-1) P'(z,s) + \left[-\lambda_1 z^2 (1 + \varepsilon N) + z\{s + \mu_1 + \beta + \xi + \lambda_1 (1 + \varepsilon N)\} - \mu_1\right] P(z,s)$$
$$- \alpha\, z\, Q(z,s) = \mu_1 (z-1)\, \bar{P}_0(s) + z^{M+1} (1-z)\lambda_1(M)\, \bar{P}_M(s) + z + \xi z \sum_{n=0}^{M} \bar{P}_n(s) \tag{17}$$

Similarly, from equations (11)–(13) and using (14)–(16), we have

$$\beta z P(z,s) + [\mu_2 - z(s+\mu_2+\alpha+\xi)]Q(z,s) - \mu_2(1-z)\overline{Q}_0(s) + \xi z \sum_{n=0}^{M} \overline{Q}_n(s) = 0 \tag{18}$$

Eliminating Q(z, s) from equations (17) and (18), we have

$$P'(z,s) + \frac{\eta_1(z)}{\eta_2(z)} P(z,s) = \frac{1}{\eta_2(z)}\left[ z_1 + z_2\overline{Q}_0(s) + z_3\overline{P}_0(s) + z_4\,\overline{P}_M(s) + z_5\sum_{n=0}^{M}\overline{P}_n(s) + z_6\sum_{n=0}^{M}\overline{Q}_n(s)\right] \tag{19}$$

Where dashes denote the differentiation of the function w. r. t. z and

$$\eta_1(z) = [a_2 a_3 z^3 + (\alpha\beta - a_1 a_3 - a_2\mu_2)z^2 + (a_1\mu_2 + a_3\mu_1)z - \mu_1\mu_2]$$

$$\eta_2(z) = z^2\,(z-1)\,(\mu_2 - a_3 z)\lambda_1\,\varepsilon$$

$$z_1 = z(\mu_2 - a_3 z)$$

$$z_2 = \alpha\mu_2 z(1-z)$$

$$z_3 = \mu_1\,(z-1)(\mu_2 - a_3 z)$$

$$z_4 = \lambda_1(M)\;z^{M+1}\,(1-z)(\mu_2 - a_3 z)$$

$$z_5 = \xi z(\mu_2 - a_3 z)$$

$$z_6 = -\alpha\,\xi\,z^2$$

$$a_1 = [s + \mu_1 + \beta + \xi + \lambda_1(1+\varepsilon N)]$$

$$a_2 = \lambda_1(1+\varepsilon N)$$

$$a_3 = [s + \mu_2 + \alpha + \xi]$$

$$\lambda_1(M) = \lambda_1[1 + \varepsilon(N-M)]$$

In equation (19), the co-efficient of P(z,s) can be re-written as:

$$\frac{\eta_1(z)}{\eta_2(z)} = \frac{A}{(z-1)} + (B - a_2/\lambda_1\varepsilon)\frac{1}{z} + \frac{C}{\lambda_1\varepsilon z^2} + \frac{D}{(\mu_2 - a_3 z)} \tag{20}$$

where,

$$A = \frac{\alpha\beta + a_1\mu_2 - a_2\mu_2 - \mu_1\mu_2 - a_1 a_3 + a_2 a_3 + a_3\mu_1}{\lambda_1\varepsilon(\mu_2 - a_3)}$$

$$B = -\left[\frac{1}{\lambda_1\varepsilon}(s+\beta+\xi)\right]$$

$$C = \mu_1 + \lambda_1 (1 + \varepsilon N)$$

$$D = -\left[ \frac{a_3 \alpha \beta}{\lambda_1 \varepsilon (s + \alpha + \xi)} \right]$$

Using equation (20) in (19) and solving the equation (19), we have

$$P(z,s) = \frac{L_1(z) + L_2(z)\overline{Q}_0(s) + L_3(z)\overline{P}_0(s) + L_4(z)\overline{P}_M(s) + L_5(z)\sum_{n=0}^{M}\overline{P}_n(s) + L_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{L(z)} \tag{21}$$

where

$$L(z) = (z-1)^A \cdot z^{\left(B - \frac{a_2}{\lambda_1 \varepsilon}\right)} \cdot (\mu_2 - a_3 z)^{-D/a_3} \cdot e^{-C/z\lambda_1\varepsilon}$$

$$L_i(z) = \int_0^Z \frac{z_i}{\eta_2(z)} L(z)\, dz\ ; \qquad i = 1, 2, 3, 4, 5, 6.$$

Putting the value of P(z,s) in equation (18), and on simplification, we have

$$Q(z,s) = \frac{L_7(z) + \overline{Q}_0(s)L_8(z) + \overline{P}_0(s)L_9(z) + \overline{P}_M(s)L_{10}(z) + \sum_{n=0}^{M}\overline{P}_n(s)L_{11}(z) + \sum_{n=0}^{M}\overline{Q}_n(s)L_{12}(z)}{B(z)L(z)} \tag{22}$$

where

$$B(z) = \mu_2 - z(s + \mu_2 + \alpha + \xi)$$

$$L_7(z) = -\beta z L_1(z)$$

$$L_8(z) = -\beta z L_2(z) + \mu_2 (1 - z)\, L(z)$$

$$L_9(z) = -\beta z L_3(z)$$

$$L_{10}(z) = -\beta z L_4(z)$$

$$L_{11}(z) = -\beta z\, L_5(z)$$

$$L_{12}(z) = -\beta z\, L_6(z)$$

Adding equations (21) and (22), we have

$$R(z,s) = \frac{C_1(z) + C_2(z)\overline{Q}_0(s) + C_3(z)\overline{P}_0(s) + C_4(z)\overline{P}_M(s) + C_5(z)\sum_{n=0}^{M}\overline{P}_n(s) + C_6(z)\sum_{n=0}^{M}\overline{Q}_n(s)}{B(z)\, L(z)} \tag{23}$$

where

Ci (z) = B (z) Li (z) + Li+6 (z) ; i = 1, 2, 3, 4, 5, 6

Relation (23) is a polynomial in z and exists for all values of z, including the three zeros of the denominator $z_1, z_2$ and $z_3$ (say).

where,

$$z_1 = 1$$

$$z_2 = \frac{\mu_2}{a_3}$$

$$z_3 = \frac{\mu_2}{s + \mu_2 + \alpha + \xi}$$

The unknown quantities $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ are obtained by setting the numerator equal to zero on substituting the three zeros z1 , z2 , z3 of the denominator (at each of which the numerator must vanish). Also the remaining quantities $\sum_{n=0}^{M} \overline{P}_n(s)$ and $\sum_{n=0}^{M} \overline{Q}_n(s)$ are obtained by setting z=1 in equations (17) and (18) respectively, thus we have.

$$P(1,s) = \sum_{n=0}^{M} \overline{P}_n(s) = \frac{s+\alpha}{s(s+\alpha+\beta)}$$

$$Q(1,s) = \sum_{n=0}^{M} \overline{Q}_n(s) = \frac{\beta}{s(s+\alpha+\beta)}$$

After simplification, the three equations determining the unknown quantities $\overline{Q}_0(s), \overline{P}_0(s)$ and $\overline{P}_M(s)$ are:

$$L_2(z_1)\overline{Q}_0(s) + L_3(z_1)\overline{P}_0(s) + L_4(z_1)\overline{P}_M(s) = -\left\{L_1(z_1) + L_5(z_1)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_1)\frac{\beta}{s(s+\alpha+\beta)}\right\} \quad (24)$$

$$L_2(z_2)\overline{Q}_0(s) + L_3(z_2)\overline{P}_0(s) + L_4(z_2)\overline{P}_M(s) = -\left\{L_1(z_2) + L_5(z_2)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_2)\frac{\beta}{s(s+\alpha+\beta)}\right\} \quad (25)$$

$$L_2(z_3)\overline{Q}_0(s) + L_3(z_3)\overline{P}_0(s) + L_4(z_3)\overline{P}_M(s) = -\left\{L_1(z_3) + L_5(z_3)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_3)\frac{\beta}{s(s+\alpha+\beta)}\right\} \quad (26)$$

We can re-write equations (24)-(26) as

$$X_2\overline{Q}_0(s) + X_3\overline{P}_0(s) + X_4\overline{P}_M(s) = -X_1 \quad (27)$$

where

$$X_i = \begin{bmatrix} L_i(z_1) \\ L_i(z_2) \\ L_i(z_3) \end{bmatrix} \quad ; \quad i = 2, 3, 4$$

and

$$X_1 = \begin{bmatrix} L_1(z_1) + L_5(z_1)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_1)\frac{\beta}{s(s+\alpha+\beta)} \\ L_1(z_2) + L_5(z_2)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_2)\frac{\beta}{s(s+\alpha+\beta)} \\ L_1(z_3) + L_5(z_3)\frac{s+\alpha}{s(s+\alpha+\beta)} + L_6(z_3)\frac{\beta}{s(s+\alpha+\beta)} \end{bmatrix}$$

On solving these equations, we have

$$\overline{Q}_0(s) = \frac{|\Delta_1|}{|\Delta_0|}$$

$$\overline{P}_0(s) = \frac{|\Delta_2|}{|\Delta_0|}$$

$$\overline{P}_M(s) = \frac{|\Delta_3|}{|\Delta_0|}$$

where

$\Delta_0 = (X_2, X_3, X_4)$ and

$\Delta_i = \Delta_0$, with its ith column replaced by $(-X_1)$

## 4. STEADY STATE SOLUTION

This can at once be obtained by using the well-known property of the Laplace transform given below:

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,\bar{f}(s)$, If the limit on the left hand side exists.

By employing this property, we have from equation (23).

$$R(z) = \frac{Q_0 K_1(z) + K_2(z)P_0 + K_3(z)P_M + K_4(z)\sum_{n=0}^{M} P_n + K_5(z)\sum_{n=0}^{M} Q_n + K(z)}{B_1(z)\, L''(z)} \tag{28}$$

where,

$R(z) = \sum_{n=0}^{M} R_n z^n$,

$$R_n = \lim_{s\to 0} s\,\overline{R}_n(s)$$ and

$$B_1(z) = B(z)\big|_{s=0}$$

$$L''(z) = L(z)\big|_{s=0}$$

$$K_i(z) = B_1(z)L'_{i+1}(z) + L'_{i+7}(z)\Big|_{s=0} \quad ;\ i = 1, 2, 3, 4, 5.$$

$$L_j(z) = \int \left[\frac{z_j}{\eta_2(z)} L(z)\right]_{\varepsilon=0} dz \quad ;\ j = 2, 3, 4, 5, 6.$$

$$L'_8(z) = -\beta z L'_2(z) + \mu_2(1-z)L''(z)$$

$$L'_9(z) = -\beta z L'_3(z)$$

$$L'_{10}(z) = -\beta z L'_4(z)$$

$$L'_{11}(z) = -\beta z L'_5(z)$$

$$L'_{12}(z) = -\beta z L'_6(z)$$

$$K(z) = \lim_{s\to 0} s\, C_1(z)$$

The unknown quantities Q0, P0, PM, $\sum_{n=0}^{M} P_n$ and $\sum_{n=0}^{M} Q_n$ can be evaluated as before.

## 5. PARTICULAR CASES

Case (a) Setting $\varepsilon$ =0 or n=N in equations (17) and (18), (i.e., when the arrival rate in the environmental state E is $\lambda_1$, a constant), we have

$$X_1(z)P(z,s) + X_2(z)Q(z,s) + X_3(z) = 0 \tag{29}$$

$$X_4(z)P(z,s) + X_5(z)Q(z,s) + X_6(z) = 0 \tag{30}$$

where

$$X_1(z) = -\left[\lambda_1 z^2 - z\,(s + \lambda_1 + \mu_1 + \beta + \xi) + \mu_1\right]$$

$$X_2(z) = -\alpha z$$

$$X_3(z) = -\left[\mu_1(z-1)\,\overline{P}_0(s) + \lambda_1 z^{M+1}(1-z)\overline{P}_M(s) + z + \xi z \sum_{n=0}^{M} \overline{P}_n(s)\right]$$

$$X_4(z)=\beta z$$

$$X_5(z)=\left[\mu_2 - z(s+\mu_2+\alpha+\xi)\right]$$

$$X_6(z)=\left[\mu_2(z-1)\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\right]$$

From equations (29) and (30), we have.

$$P(z,s)=\frac{X_2(z)X_6(z)-X_3(z)X_5(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \tag{31}$$

$$Q(z,s)=\frac{X_4(z)X_3(z)-X_1(z)X_6(z)}{X_1(z)X_5(z)-X_2(z)X_4(z)} \tag{32}$$

Thus, on adding equations (31) and (32), we have

$$R(z,s)=\frac{\begin{array}{l}\mu_2(z-1)\left[X_2(z)-X_1(z)\right]\overline{Q}_0(s)+\xi z\sum_{n=0}^{M}\overline{Q}_n(s)\left[X_2(z)-X_1(z)\right]+\mu_1(1-z)\\ \left[X_4(z)-X_5(z)\right]\overline{P}_0(s)+\lambda_1 z^{M+1}\left[X_5(z)-X_4(z)\right](1-z)\overline{P}_M(s)+\\ z\left[X_5(z)-X_4(z)\right]+\xi z\sum_{n=0}^{M}\overline{P}_n(s)\left[X_5(z)-X_4(z)\right]\end{array}}{-z^2 s^2+s X_7(z)+(1-z)X_8(z)-z^2\xi(\alpha+\beta+\xi)} \tag{33}$$

where

$$X_7(z)=\lambda_1 z^3-z^2(\lambda_1+\mu_1+\mu_2+\alpha+\beta+2\xi)+z(\mu_1+\mu_2)$$

$$X_8(z)=-z^2\lambda_1(\alpha+\mu_2+\xi)+z\left[\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\right]-\mu_1\mu_2$$

Relation (33) being a polynomial in z exists for all values of z, including the three zeros of the denominator. The unknown quantities $\overline{P}_0(s)$, $\overline{Q}_0(s)$ and $\overline{P}_M(s)$ can be obtained by setting the numerator equal to zero on substituting the three zeros, α1, α2 and α3 (say) of the denominator (at each of which the numerator must vanish).

The remaining quantities $\sum_{n=0}^{M}\overline{P}_n(s)$ and $\sum_{n=0}^{M}\overline{Q}_n(s)$ are obtained by setting z=1, in equations (31) and (32) respectively, thus we have

$$P(1,s)=\sum_{n=0}^{M}\overline{P}_n(s)=\frac{s+\alpha}{s(s+\alpha+\beta)}$$

$$Q(1,s)=\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{\beta}{s(s+\alpha+\beta)}$$

and $$\sum_{n=0}^{M}\overline{R}_n(s)=\sum_{n=0}^{M}\overline{P}_n(s)+\sum_{n=0}^{M}\overline{Q}_n(s)=\frac{1}{s}$$

Case (b) Now letting α→∞, β→0 and setting μ1= μ2= μ (say) in relation (33), we have

$$R(z,s)=\frac{(1-z)\,\mu\overline{R}_0(s)-(1-z)\lambda_1 z^{M+1}\,\overline{P}_M(s)-z-\xi z/s}{\lambda_1 z^2-z(s+\lambda_1+\mu+\xi)+\mu} \quad (34)$$

where

$$\overline{R}_0(s)=\overline{P}_0(s)+\overline{Q}_0(s)$$

$$R(z,s)=\lim_{\beta\to 0}\left\lfloor \lim_{\alpha\to\infty} R(z,s)\right\rfloor$$

Relation (34) is a polynomial in z and therefore exists for all values of z, including the two zeros of the denominator. Hence, the unknown quantities $\overline{R}_0(s)$ and $\overline{P}_M(s)$ can be evaluated as before.

**Steady State Solution:**

Case (a) Relation (33), on applying the theory of Laplace transforms gives the steady state form

$$R(z)=\frac{\begin{array}{l}\mu_2(1-z)\{\alpha z+z(\lambda_1+\mu_1+\beta+\xi)-\lambda_1 z^2-\mu_1\}Q_0\\ +\mu_1(1-z)[\beta z-\{\mu_2-z(\mu_2+\alpha+\xi)\}]P_0+\lambda_1 z^{M+1}(1-z)\{\mu_2-z(\mu_2+\alpha+\xi)-\beta z\}P_M\\ +\{\xi z/(\alpha+\beta+\xi)\}[\beta\{\lambda_1 z^2-z(\lambda_1+\mu_1+\alpha+\beta+\xi)+\mu_1\}+(\alpha+\xi)\{\mu_2-z(\mu_2+\alpha+\beta+\xi)\}]\end{array}}{\begin{array}{l}z^3\lambda_1(\mu_2+\alpha+\xi)-z^2[\lambda_1(\mu_2+\alpha+\xi)+\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\xi(\alpha+\beta+\xi)]\\ +z[\{\alpha\mu_1+\mu_2(\lambda_1+\mu_1+\beta+\xi)\}+\mu_1\mu_2]-\mu_1\mu_2\end{array}} \quad (35)$$

where

$$R(z)=\lim_{s\to 0} s\; R(z,s)$$

We can re- write equation (35) as

$$R(z)=\frac{T(z)Q_0+N(z)P_0+L(z)P_M+M(z)}{K(z)} \quad (36)$$

Where T(z), N(z) and L(z) are the co-efficient of Q0, P0 and PM respectively in the numerator of equation (35) and K(z) is the denominator of equation (35).

Equation (36) is a polynomial in z and exists for all values of z, including three zeros of the denominator. The unknown quantities Q0, P0 and PM are obtained by setting the numerator equal to zero on substituting the three zeros b1, b2 and b3 (say) of the denominator (at each of which the numerator must vanish).

The three equations determining the unknown quantities Q0, P0 and PM are:

$$T(b_1)Q_0+N(b_1)P_0+L(b_1)P_M=-M(b_1) \quad (37)$$

$$T(b_2)Q_0 + N(b_2)P_0 + L(b_2)P_M = -M(b_2) \tag{38}$$

$$T(b_3)Q_0 + N(b_3)P_0 + L(b_3)P_M = -M(b_3) \tag{39}$$

After solving these equations, we have

$$Q_0 = \frac{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}}{A}$$

$$P_0 = \frac{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}}{A}$$

$$P_M = \frac{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}}{A}$$

where

$$A = \begin{vmatrix} T(b_1) & N(b_1) & L(b_1) \\ T(b_2) & N(b_2) & L(b_2) \\ T(b_3) & N(b_3) & L(b_3) \end{vmatrix}$$

Aij is the co-factor of the (i, j)th element of A.By putting the values of Q0, P0 and PM in equation (36), we have

$$R(z) = \frac{\begin{array}{l} T(z)[-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}] + N(z)[M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}] \\ + L(z)[-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}] + A \cdot M(z) \end{array}}{A \cdot K(z)} \tag{40}$$

## 6. MEAN QUEUE LENGTH

Define, Lq= Expected number of customers in the queue including the one in service. Then Lq = $R'(z)\big|_{z=1}$

Therefore, from equation (40), we have

$$L_q = \frac{\begin{array}{l} K(1)[T'(1)\{-M(b_1)A_{11} + M(b_2)A_{21} - M(b_3)A_{31}\} + N'(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} \\ + L'(1)\{-M(b_1)A_{13} + M(b_2)A_{23} - M(b_3)A_{33}\} + A \cdot M'(1)] - [T(1)\{-M(b_1)A_{11} + M(b_2)A_{21} \\ - M(b_3)A_{31}\} + N(1)\{M(b_1)A_{12} - M(b_2)A_{22} + M(b_3)A_{32}\} + L(1)\{-M(b_1)A_{13} + M(b_2)A_{23} \\ - M(b_3)A_{33}\} + A \cdot M(1)]\, K'(1) \end{array}}{A \cdot [K(1)]^2} \tag{41}$$

where dashes denotes the first derivative with respect to z.

Case (b) Relation (34), on applying the theory of Laplace transforms gives the steady state form

$$R(z) = \frac{(1-z)\mu R_0 - (1-z)\lambda_1 z^{M+1} P_M - \xi z}{\lambda_1 z^2 - z(\lambda_1 + \mu + \xi) + \mu} \tag{42}$$

where

$$R(z)=\lim_{s\to 0} s\ R(z,s)$$

Equation (42) being a polynomial in z exists for all values of z, including the two zeros of the denominator. Hence, the unknown quantities R0 and PM are obtained by setting the numerator equal to zero on substituting the two zeros a1 and a2 (say) of the denominator (at each of which the numerator must vanish).

Two equations determining the constants R0 and PM are:

$$(1-a_1)\mu R_0-(1-a_1)\lambda_1 a_1^{M+1} P_M=\xi a_1 \tag{43}$$

$$(1-a_2)\mu R_0-(1-a_2)\lambda_1 a_2^{M+1} P_M=\xi a_2 \tag{44}$$

On solving these equations, we have

$$P_M=\frac{(a_1-a_2)}{a_1^{M+1}-a_2^{M+1}} \quad \text{and} \quad R_0=\frac{a_2\xi}{(1-a_2)\mu}+\frac{\lambda_1}{\mu}a_2^{M+1}\frac{a_1-a_2}{a_1^{M+1}-a_2^{M+1}};$$

where

$$\lambda_1(1-a_1)\ (1-a_2)=-\xi$$

Now, from equation (42), we have

$$R(z)=\frac{\xi+\lambda_1(1-z)(1-a_2)\dfrac{(a_1-a_2)}{a_1^{M+1}-a_2^{M+1}}\ \dfrac{a_2^{M+1}-z^{M+1}}{a_2-z}}{\lambda_1(z-a_1)\ (a_2-1)} \tag{45}$$

$$=\frac{1}{\lambda_1 a_1(1-a_2)}\left[\xi+\lambda_1(1-z)(1-a_2)P_M+\{a_2^M+a_2^{M-1}z+...+z^M\}\right]\sum_{i=0}^{\infty}\left(\frac{z}{a_1}\right)^i$$ Rn = The co-efficient of zn

$$R_n=\frac{a_2}{\mu(1-a_2)}\left[\xi(1-P_M)+a_2^{M-n}\frac{a_1^{n+1}-a_2^{n+1}}{a_1-a_2}\right]\left(\frac{\lambda_1}{\mu}\right)^n a_2^n \tag{46}$$

If $\xi=0$ (i.e., no catastrophe is allowed), then from relation (42), we have

$$R(z)=\frac{\mu R_0-\lambda_1 z^{M+1}P_M}{\mu-\lambda_1 z} \tag{47}$$

The condition, $\lim_{z\to 1} R(z)=1$ gives

$$\mu R_0-\lambda_1 P_M=\mu-\lambda_1 \tag{48}$$

As R(z) is analytic, the numerator and denominator of equation (47) must vanish simultaneously for z= $\mu/\lambda 1$, which is a zero of its denominator. Equating the numerator of equation (47) to zero for z= $\mu/\lambda 1$ we have

$$R_0 = \rho^{-M} P_M \tag{49}$$

Relation (48) and (49) gives

$$R_0 = \frac{1-\rho}{1-\rho^{M+1}} \quad , \quad P_M = \frac{(1-\rho)\rho^M}{1-\rho^{M+1}}$$

Now, from equation (47), we have

$$R(z) = \frac{1-\rho}{1-\rho^{M+1}} \quad . \quad \left[\frac{1-(\rho\, z)^{M+1}}{1-\rho z}\right] \tag{50}$$

Which is a well known result of the M/M/1 queue with finite waiting space M.

When there is an infinite waiting space, the corresponding expression for R(z) is obtained by letting M tends to infinity in equation (50), If Max ($\rho$, $|z|$) <1.

$$R(z) = \frac{1-\rho}{1-\rho z} \tag{51}$$

Which is again a well known result of the M/M/1 queue with infinite waiting space.

## 7. APPLICATIONS OF THE MODEL

1. In nature, there are many creatures such as cockroaches, ants, mosquitoes etc whose movement is restricted with the change of temperature (environment). As the temperature drops below a critical temperature say $T_0$, the movement (production) of such like creatures becomes almost zero. On the other hand, as the temperature goes higher than $T_0$ the movement becomes normal. The catastrophes may occur with these creatures in both the environmental states i.e., spray etc which make them zero instantaneously. Then the number of such like creatures present in any area can be estimated by using the described queueing model with environmental change and catastrophes.
2. In agriculture, if a crop is infected with a particular species of insects due to change in temperature (environment), we may use some chemical agents or compounds to treat such type of insects. The number of bacteria that destroys the crop, in large part, relies on the effectiveness and amount of the chemical reagents used. In other words, the use of the chemical reagents can wipe out the whole of the insects or a part of it. The effect of these chemical reagents on bacteria which make them zero instantaneously can be regarded as the occurrence of a catastrophe.

## 8. CONCLUSIONS

In the present paper, we have established a queueing system with catastrophe, state dependent input parameter and environmental change. We have also obtained some interesting particular cases with (without) catastrophe and steady state results in detail.

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